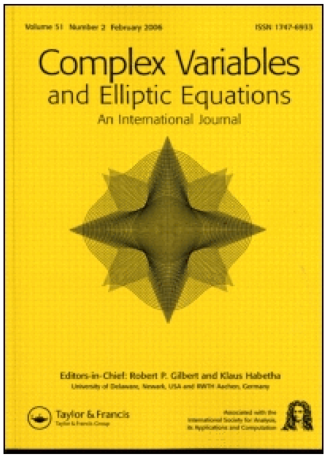

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On a class of singular integral equations with the linear fractional Carleman shift and the degenerate kernel

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On a class of singular integral equations with the linear fractional Carleman shift and the degenerate kernel

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This article deals with the solvability, the explicit solutions of a class of singular integral equations with a linear-fractional Carleman shift and the degenerate kernel on the unit circle by means of the Riemann boundary value problem and of a system of linear algebraic equations. All cases about index of the coefficients in the equations are considered in detail.

Keywords: Integral operators; Singular Integral equations; Riemann boundary value problems

AMS Subject Classifications: 47G05; 45G05; 45E05

1. Introduction

Singular integral equations with a shift (SIES) have been studied for a long time (see [1,2] and references therein). Many papers devoted to singular integral operators with a shift (SIOS) are given to the construction of the Fredholm theory. Once M. G. Krein called the Fredholm theory of linear operators a *rough* theory, and the theory describing its defect subspaces a *delicate* theory [3]. However, the Fredholm theory of these operators brings about only one thing, the defect of dimensions of kernel of the operator and its dual operator. In other words, it is only the defect of the numbers of linear independent solutions of the homogeneous equations reduced by the operator and the corresponding dual operator. So the question of solving (and even of estimating the numbers of solutions) of the corresponding equations actually remains open [4]. There are only a few special types of SIES for which it is possible to answer this question to some extent [2,5]. Among the SIES of this type not reducible to two-term boundary value problems, the most general and important is the class of singular integral equations with a

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linear-fractional Carleman shift. In our view, the singular integral equations with a linear-fractional Carleman shift in the unit circle, in addition, deserve the interest. Factorization is the main method used by some authors to investigate Fredholm and solvability theory for SIES (see [3,4] and references therein). In [6], two of us gave a general formula of linear-fractional Carleman shifts on the unit circle and solved by means of Riemann boundary value problem for a class of singular integral equations with a linear-fractional Carleman shift on the unit circle. In this article, we study the solvability for a class of singular integral equations with a linear-fractional Carleman shift and with the degenerate kernels on the unit circle. In genaral, one knows that the singular integral operator of Cauchy's type (denoted by S) do not commute with Carleman shift opetaor (denoted by W), but the difference beetwen them $WS-SW$ is a compact operator [2]). In section 2, we obtain some identities relating to those operators. The scheme of our investigation is divided into two parts: first, we move the degenerate kernels to the right-side hand of the equation. Based on the identity $W^n=I$, we construct the orthogonal projectors and reduce the equation to a system of singular integral equations without shift and solve this system by means of Riemann boundary value problem. Second, we reconstruct the solution of the orginal equation from the solutions of system of equations that can be solved, but its solution depends on some unknown parameters. As indicated below, the equations of the type (1.1) can be solved by means of Riemann boundary value problem and by of a system of linear algebraic equations.

Let $\Gamma=\{t\in\mathbb{C}, |t|=1\}$ be the unit circle on the complex plane $\mathbb{C}$ and let $X:=H^{\mu}(\Gamma)$, $0<\mu<1$. Let

$$\omega(t)=\frac{\alpha t+\beta}{\gamma t+\delta}, \quad (\alpha\delta-\beta\gamma=1,\ \gamma\neq 0)$$

be a Carleman linear-fractional function shift of order of n on Γ, i.e.,

$$\begin{cases} \omega:\Gamma\to\Gamma, \\ \omega_n(t)=t, \quad \text{for every } t\in\Gamma\ (\omega_k(t):=\omega(\omega_{k-1}(t)),\ \omega_0(t)\equiv t), \\ \omega_k(t)\not\equiv t, \quad \text{for some } t\in\Gamma,\ k=1,2,\ldots,n-1, \\ \omega \text{ positive orientation of } \Gamma \end{cases}$$

The such functions $\omega(t)$ might be of the form

$$\omega(t)=\begin{cases} \dfrac{t-\alpha}{\bar{\alpha}t-1}, & \text{if} \quad n=2; \\ e^{i\theta}\dfrac{t-\alpha}{\bar{\alpha}t-1}, & \text{if } n>2, \end{cases}$$

where $|\alpha|<1$, $\cos\theta=1-2(1-\alpha\bar{\alpha})\cos^2(k\pi/n)$, for some $k\in\{1,2,\ldots,n-1\}$, $(k,n)=1$ [6].

In this article, we study the solvability of the equations of the form

$$a(t)\varphi(t)+\frac{b(t)}{n}\sum_{k=0}^{n-1}\varepsilon_{\ell}^{n-k}\frac{1}{\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau-\omega_k(t)}\mathrm{d}\tau + \sum_{j=1}^{m}\frac{1}{\pi i}\int_{\Gamma}a_j(t)b_j(\tau)\varphi(\tau)\mathrm{d}\tau=f(t), \tag{1.1}$$

where $1\leq\ell\leq n-1, \varepsilon_1=e^{(2\pi i/n)}, \ \varepsilon_{\ell}=\varepsilon_1^{\ell}$.

2. Some identities of singular integral operator of Cauchy's type and linear-fractional shift operator on unit circle

Consider the following operators in X:

$$\begin{aligned}(S\varphi)(t) &= \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\mathrm{d}\tau,\\ (W\varphi)(t) &= \varphi(\omega(t)),\\ P_k &= \frac{1}{n}\sum_{j=1}^{n} \varepsilon_k^{n-1-j} W^{j+1}, \quad k = 1, 2, \ldots.\end{aligned} \tag{2.1}$$

In the sequel, we shall need the following identities [7,8]:

$$\begin{cases} W^k = \sum_{j=1}^{n} \varepsilon_j^k P_j, \quad k = 1, 2, \ldots, n,\\ P_k P_j = \delta_{kj} P_j, \quad k, j = 1, 2, \ldots, n,\\ \sum_{j=1}^{n} P_j = I, \end{cases} \tag{2.2}$$

where δ_{kj} is the Kronecker symbol. For every $a \in X$ we write $(K_a\varphi)(t) = a(t)\varphi(t)$.

Lemma 2.1 *Let $a \in X$ be fixed. Then for every (k, j), k, $j \in \{1, 2, \ldots, n\}$ there exists an element $b \in X$ such that $K_b X_j \subset X_k$* and *$P_k K_a P_j = K_b P_j$, where $x_k := P(X)$. The Function $b(t)$ will be denoted by $a_{kj}(t)$ and determined as follows*

$$a_{kj}(t) = \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_{\nu+1}^{j-k} a(\omega_{\nu+1}(t)). \tag{2.3}$$

Proof By using (2.2) we get

$$\begin{aligned} P_k K_a P_j &= \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_k^{n-1-\nu} W^{\nu+1} K_a P_j = \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_k^{n-1-\nu} a(\omega_{\nu+1}(t)) W^{\nu+1} P_j\\ &= \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_k^{-(\nu+1)} a(\omega_{\nu+1}(t)) \sum_{\mu=1}^{n} \varepsilon_\mu^{\nu+1} P_\mu P_j = \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_k^{-(\nu+1)} a(\omega_{\nu+1}(t)) \varepsilon_j^{\nu+1} P_j\\ &= \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_{\nu+1}^{-k} \varepsilon_{\nu+1}^{j} a(\omega_{\nu+1}(t)) P_j = \left(\frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_{\nu+1}^{j-k} a(\omega_{\nu+1}(t))\right) P_j = a_{kj}(t) P_j, \end{aligned}$$

where

$$a_{kj}(t) = \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_{\nu+1}^{j-k} a(\omega_{\nu+1}(t)). \tag{2.4}$$

Putting $a_{kj}(t) := b(t)$, we obtain $b \in X$ and $P_k K_a P_j = K_b P_j$. ■

Lemma 2.2 *Let $a \in X$ be fixed. Then for any $k, j \in \{1, 2, \ldots, \mathrm{n}\}$, we have*

$$P_k K_{a_{kj}} = K_{a_{kj}} P_j,$$

where $a_{kj}(t)$ *are determined by* (2.4).

Proof For any $\varphi \in X$ we have

$$(P_k K_{a_{kj}}\varphi)(t) = P_k a_{kj}(t)\varphi(t) = \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_k^{n-1-\nu} W^{\nu+1} a_{kj}(t)\varphi(t)$$

$$= \left(\frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_k^{-(\nu+1)} W^{\nu+1}\right)\left(\frac{1}{n}\sum_{\mu=1}^{n} \varepsilon_{\mu+1}^{j-k} a(\omega_{\mu+1}(t))\right)\varphi(t)$$

$$= \frac{1}{n}\sum_{\nu=1}^{n}\left[\frac{1}{n}\sum_{\mu=1}^{n} \varepsilon_{\mu+1}^{j-k}\varepsilon_{\nu+1}^{j-k} a(\omega_{\mu+1+\nu+1}(t))\right]\varepsilon_{\nu+1}^{k-j}\varepsilon_k^{-(\nu+1)}\big(W^{\nu+1}\varphi\big)(t)$$

$$= \frac{1}{n}\sum_{\nu=1}^{n} a_{kj}(t)\varepsilon_{\nu+1}^{k-j}\varepsilon_{\nu+1}^{-k}\big(W^{\nu+1}\varphi\big)(t)$$

$$= a_{kj}(t)\left[\frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_j^{n-1-\nu} W^{\nu+1}\right]\varphi(t) = a_{kj}(t)(P_j\varphi)(t) = (K_{a_{kj}}P_j)(t).$$

Thus $P_k K_{a_{kj}} \equiv K_{a_{kj}} P_j$. ■

LEMMA 2.3 *Let* $\varphi \in X$. *Then for every* $z \in \mathbb{C}$ *we have*

(1) $(SW^k\varphi)(z) = (W^k S\varphi)(z) - (W^{k-1}S\varphi)(\alpha/\gamma)$, $k = 1, 2, \ldots, W^0 = I$.
(2) $(P_k S\varphi)(z) = (SP_k\varphi)(z) + (1/\varepsilon_k - 1)(SP_k\varphi)(\alpha/\gamma)$, $k = 1, 2, \ldots, n-1$,

where α *and* γ *are the coefficients of the linear-fractional function* $\omega(z) = ((\alpha z + \beta)/(\gamma z + \delta))$.

Proof

(1) By induction on k. For $k = 1$ we have

$$(SW\varphi)(z) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\omega(\tau))}{\tau - z}\,\mathrm{d}\tau.$$

Put

$$\tau = \omega^{-1}(x) = \frac{\delta x - \beta}{-\gamma x + \alpha}, \quad \text{then } \mathrm{d}\tau = \frac{1}{(-\gamma x + \alpha)^2}\,\mathrm{d}x.$$

Therefore,

$$(SW\varphi)(z) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(x)(1/(\gamma x - \alpha)^2)}{(\delta x - \beta/-\gamma x + \alpha) - z}\,\mathrm{d}x$$

$$= \frac{1}{\pi i}\int_\Gamma \frac{\varphi(x)}{(-\gamma x + \alpha)(\delta x - \beta - z(-\gamma x + \alpha))}\,\mathrm{d}x$$

$$= \frac{1}{\pi i}\int_\Gamma \left(\frac{1}{x - (\alpha z + \beta/\gamma z + \delta)} - \frac{1}{x - (\alpha/\gamma)}\right)\varphi(x)\mathrm{d}x$$

$$= \frac{1}{\pi i}\int_\Gamma \frac{\varphi(x)}{x - \omega(z)}\,\mathrm{d}x - \frac{1}{\pi i}\int_\Gamma \frac{\varphi(x)}{x - (\alpha/\gamma)}\,\mathrm{d}x = (WS\varphi)(z) - (S\varphi)\left(\frac{\alpha}{\gamma}\right).$$

So, we obtain

$$(SW\varphi)(z) = (WS\varphi)(z) - (S\varphi)\left(\frac{\alpha}{\gamma}\right), \quad \text{for any } z \in \mathbb{C}. \tag{2.5}$$

Suppose that (i) is true for $k = m$. For $k = m+1$ we find

$$\begin{aligned}
(SW^{m+1}\varphi)(z) &= [SW(W^m\varphi)](z) \\
&= [WS(W^m\varphi)](z) - [S(W^m\varphi)]\left(\frac{\alpha}{\gamma}\right) = W[(SW^m\varphi)(z)] - (SW^m\varphi)\left(\frac{\alpha}{\gamma}\right) \\
&= W\left[(W^mS\varphi)(z) - (W^{m-1}S\varphi)\left(\frac{\alpha}{\gamma}\right)\right] - \left[(W^mS\varphi)\left(\frac{\alpha}{\gamma}\right) - (W^{m-1}S\varphi)\left(\frac{\alpha}{\gamma}\right)\right] \\
&= (W^{m+1}S\varphi)(z) - W\left[(W^{m-1}S\varphi)\left(\frac{\alpha}{\gamma}\right)\right] - (W^mS\varphi)\left(\frac{\alpha}{\gamma}\right) + (W^{m-1}S\varphi)\left(\frac{\alpha}{\gamma}\right).
\end{aligned}$$

Hence $W[(W^{m-1}S\varphi)(\alpha/\gamma)] = (W^{m-1}S\varphi)(\alpha/\gamma)$, provided $(W^{m-1}S\varphi)(\alpha/\gamma)$ is a constant. Therefore

$$(SW^{m+1}\varphi)(z) = (W^{m+1}S\varphi)(z) - (W^mS\varphi)\left(\frac{\alpha}{\gamma}\right).$$

The first part of the lemma is proved.

(2) Rewrite the equality in (1) in the form

$$(W^kS\varphi)(z) = (SW^k\varphi)(z) + (W^{k-1}S\varphi)\left(\frac{\alpha}{\gamma}\right).$$

We find

$$\begin{aligned}
(P_kS\varphi)(z) &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_k^{n-i-1}(W^{i+1}S\varphi)(z) = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_k^{n-i-1}\left[(SW^{i+1}\varphi)(z) + (W^iS\varphi)\left(\frac{\alpha}{\gamma}\right)\right] \\
&= \left[S\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_k^{n-i-1}W^{i+1}\right)\varphi\right](z) + \frac{1}{\varepsilon_k}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_k^{n-i}W^i\right)S\varphi\right]\left(\frac{\alpha}{\gamma}\right) \\
&= (SP_k\varphi)(z) + \frac{1}{\varepsilon_k}(P_kS\varphi)\left(\frac{\alpha}{\gamma}\right).
\end{aligned} \tag{2.6}$$

Substituting $z = (\alpha/\gamma)$ in formula (2.6), we get

$$\left(1 - \frac{1}{\varepsilon_k}\right)(P_kS\varphi)\left(\frac{\alpha}{\gamma}\right) = (SP_k\varphi)\left(\frac{\alpha}{\gamma}\right). \tag{2.7}$$

If $k \in \{1, 2, \ldots, n-1\}$ then $\varepsilon_k \neq 1$. In this case, substituting (2.7) in (2.6) we receive

$$(P_kS\varphi)(z) = (SP_k\varphi)(z) + \frac{1}{\varepsilon_k - 1}(SP_k)\varphi\left(\frac{\alpha}{\gamma}\right). \qquad \blacksquare$$

Comment From the identity (2.5), one can say that the operators S and W do not commute to each other, but the difference of WS and SW at the a function $\varphi(t)$ always equals $(S\varphi)(\alpha/\gamma)$.

3. Reducing equation (1.1) to a system of singular integral equations

We now represent the equation (1.1) in the following form

$$a(t)\varphi(t) + b(t)(P_\ell S\varphi)(t) + \sum_{j=1}^{m} \frac{1}{\pi i}\int_\Gamma a_j(t)b_j(\tau)\varphi(\tau)\mathrm{d}\tau = f(t), \tag{3.1}$$

where $a, b, f, a_1, \ldots, a_m, b_1, \ldots, b_m \in X$ are given, and S, P_ℓ $(1 \le \ell \le n-1)$ are the operators defined by (2.1). Suppose that $a(t)$ is a non-vanishing function on Γ. Denote by $M_{b_j}, j = 1, \ldots, m,$ the linear functionals on X defined as follows

$$M_{b_j}(\varphi) = \frac{1}{\pi i}\int_\Gamma b_j(\tau)\varphi(\tau)\mathrm{d}\tau, \quad \text{for any } \varphi \in X.$$

Put $\lambda_j = M_{b_j}(\varphi)$, $j = 1, \ldots,$ m. We reduce equation (3.1) to the following problem: Find solutions φ of equation

$$a(t)\varphi(t) + b(t)(P_\ell S\varphi)(t) = f(t) - \sum_{j=1}^{m} \lambda_j a_j(t) \tag{3.2}$$

depended on the parameters $\lambda_1, \ldots, \lambda_m$, and fulfilled the following conditions

$$M_{b_j}(\varphi) = \lambda_j, \; j = 1, \ldots. \tag{3.3}$$

LEMMA 3.1 *Let $\varphi \in X$. Then φ is a solution of (3.2) if and only if $\{\varphi_k = P_k\varphi,$ $k = 1, 2, \ldots, n\}$ is a solution of the following system*

$$a^*(t)\varphi_k(t) + b^*_{k\ell}(t)(S\varphi_\ell)(t) + \frac{b^*_{k\ell}(t)}{\varepsilon_\ell - 1}(S\varphi_\ell)\left(\frac{\alpha}{\gamma}\right) = f^*_k(t), \quad k = 1, 2, \ldots, n, \tag{3.4}$$

where

$$\begin{aligned} a^*(t) &= \prod_{j=1}^{n} a(\omega^{j+1}(t)), \\ b^*_{k\ell}(t) &= \frac{1}{n}\sum_{j=1}^{n} \varepsilon_{j+1}^{\ell-k} b(\omega^{j+1}(t)) \prod_{\substack{\mu=1 \\ \mu\neq j}}^{n} a(\omega^{\mu+1}(t)), \\ f^*_k(t) &= \frac{1}{n}\sum_{j=1}^{n} \varepsilon_k^{n-1-j}\left[f(\omega^{j+1}(t)) - \sum_{\nu=1}^{m} \lambda_\nu a_\nu(\omega^{j+1}(t))\right] \prod_{\substack{\mu=1 \\ \mu\neq j}}^{n} a(\omega^{\mu+1}(t)). \end{aligned} \tag{3.5}$$

Proof Suppose that $\varphi \in X$ is a solution of (3.2). We then have

$$\prod_{\mu=1}^{n} a(\omega^{\mu+1}(t))\varphi(t) + b(t) \prod_{\substack{\mu=1 \\ \mu\neq n-1}}^{n} a(\omega^{\mu+1}(t))(P_\ell S\varphi)(t)$$
$$= \left[f(t) - \sum_{j=1}^{m} \lambda_j a_j(t) \right] \prod_{\substack{\mu=1 \\ \mu\neq n-1}}^{n} a(\omega^{\mu+1}(t)).$$

Applying the projections P_k, $k = 1, 2, \ldots, n$ to both sides of above equation and using the Lemmas 2.1 and 2.2, we obtain

$$a^*(t)(P_k\varphi)(t) + \left[\frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j+1}^{\ell-k} b(\omega^{j+1}(t)) \prod_{\substack{\mu=1 \\ \mu\neq n-1}}^{n} a(\omega^{\mu+j+2}(t)) \right] (P_\ell S\varphi)(t)$$
$$= \frac{1}{n} \sum_{j=1}^{n} \varepsilon_k^{n-j-1} \left[f(\omega^{j+1}(t)) - \sum_{\nu=1}^{m} \lambda_\nu a_\nu(\omega^{j+1}(t)) \right] \prod_{\substack{\mu=1 \\ \mu\neq n-1}}^{n} a(\omega^{\mu+j+2}(t)). \tag{3.6}$$

It is easy to see that

$$\prod_{\substack{\mu=1 \\ \mu\neq n-1}}^{n} a(\omega^{\mu+j+2}(t)) \equiv \prod_{\substack{\mu=1 \\ \mu\neq j}}^{n} a(\omega^{\mu+1}(t)) \quad \text{for any } j \in \{1, 2, \ldots, n\}.$$

Hence, (3.6) is equivalent to the following system

$$a^*(t)(P_k\varphi)(t) + b^*_{k\ell}(t)(P_\ell S\varphi)(t) = f^*_k(t), \;\; k = 1, 2, \ldots, n. \tag{3.7}$$

Using Lemma 2.3, we rewrite the system (3.7) in the form

$$a^*(t)(P_k\varphi)(t) + b^*_{k\ell}(t)(SP_\ell\varphi)(t) + \frac{b^*_{k\ell}(t)}{\varepsilon_\ell - 1}(SP_\ell\varphi)\left(\frac{\alpha}{\gamma}\right) = f^*_k(t), \;\; k = 1, 2, \ldots, n.$$

Thus $(P_1\varphi, P_2\varphi, \ldots, P_n\varphi)$ is a solution of (3.4).

Conversely, suppose that there exists $\phi \in X$ such that $(P_1\varphi, P_2\varphi, \ldots, P_n\varphi)$ is a solution of (3.4). Summing by k from 1 to n, we obtain

$$a^*(t)\,\varphi(t) + \sum_{k=1}^{n} b^*_{k\ell}(t) \left[(SP_\ell\varphi)(t) + \frac{1}{\varepsilon_\ell - 1}(SP_\ell\varphi)\left(\frac{\alpha}{\gamma}\right) \right] = \sum_{k=1}^{n} f^*_k(t). \tag{3.8}$$

From (3.5), we get

$$\sum_{k=1}^{n} b^*_{k\ell}(t) = \sum_{k=1}^{n} \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j+1}^{\ell-k} b(\omega^{j+1}(t)) \prod_{\substack{\mu=1 \\ \mu\neq j}}^{n} a(\omega^{\mu+1}(t))$$
$$= \sum_{j=1}^{n} \left[\frac{1}{n} \sum_{k=1}^{n} \varepsilon_{j+1}^{\ell-k} \right] b(\omega^{j+1}(t)) \prod_{\substack{\mu=1 \\ \mu\neq j}}^{n} a(\omega^{\mu+1}(t))$$
$$= b(t) \prod_{\substack{\mu=1 \\ \mu\neq n-1}}^{n} a(\omega^{\mu+1}(t)). \tag{3.9}$$

Similarly,

$$\sum_{k=1}^{n} f_k^*(t) = \left[f(t) - \sum_{j=1}^{m} \lambda_j a_j(t) \right] \prod_{\substack{\mu=1 \\ \mu \neq n-1}}^{n} a(\omega^{\mu+1}(t)). \tag{3.10}$$

Therefore, (3.8) is equivalent to the following equality

$$a^*(t)\varphi(t) + b(t) \prod_{\substack{\mu=1 \\ \mu \neq n-1}}^{n} a(\omega^{\mu+1}(t)) \left[(SP_\ell \varphi)(t) + \frac{1}{\varepsilon_\ell - 1} (SP_\ell \varphi)\left(\frac{\alpha}{\gamma}\right) \right]$$
$$= \left[f(t) - \sum_{j=1}^{m} \lambda_j a_j(t) \right] \prod_{\substack{\mu=1 \\ \mu \neq n-1}}^{n} a(\omega^{\mu+1}(t)).$$

This implies

$$a(t)\varphi(t) + b(t)(P_\ell S\varphi)(t) = f(t) - \sum_{j=1}^{m} \lambda_j a_j(t). \qquad \blacksquare$$

Lemma 3.2 *If* $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ *is a solution of system* (3.4) *then* $(P_1\varphi_1, P_2\varphi_2, \ldots, P_n\varphi_n)$ *is also its solution.*

Proof Suppose $(\phi_1, \phi_2, \ldots, \phi_n)$ is a solution of the system (3.4). Applying the projections P_k to both sides of k-th equation of (3.4) we get

$$a^*(t)(P_k\varphi_k)(t) + P_k \left[b_{k\ell}^*(t)(S\varphi_\ell)(t) + \frac{b_{k\ell}^*(t)}{\varepsilon_\ell - 1} (S\varphi_\ell)\left(\frac{\alpha}{\gamma}\right) \right] = P_k(f_k^*(t)). \tag{3.11}$$

It is easy to see that

$$P_k(f_k^*(t)) = f_k^*(t) \quad \text{and} \quad P_k b_{k\ell}^*(t) = b_{k\ell}^*(t) P_\ell. \tag{3.12}$$

Substituting (3.12) into (3.11), we obtain

$$a^*(t)(P_k\varphi_k)(t) + b_{k\ell}^*(t)(P_\ell S\varphi_\ell)(t) + \frac{b_{k\ell}^*(t)}{\varepsilon_\ell - 1} P_\ell\left((S\varphi_\ell)\left(\frac{\alpha}{\gamma}\right) \right) = f_k^*(t). \tag{3.13}$$

Provided $(S\varphi_\ell)(\alpha/\gamma)$ is a constant function, then

$$P_\ell\left((S\varphi_\ell)\left(\frac{\alpha}{\gamma}\right) \right) = \frac{1}{n} \sum_{j=1}^{n} \varepsilon_\ell^{n-j-1} W^{j+1}\left((S\varphi_\ell)\left(\frac{\alpha}{\gamma}\right) \right)$$
$$= \frac{1}{n} (S\varphi_\ell)\left(\frac{\alpha}{\gamma}\right) \sum_{j=1}^{n} \varepsilon_\ell^{n-j-1} = 0. \tag{3.14}$$

Using Lemma 2.3, (3.13) is equivalent to the following equation

$$a^*(t)(P_k\varphi_k)(t) + b_{k\ell}^*(t)(SP_\ell\varphi_\ell)(t) + \frac{b_{k\ell}^*(t)}{\varepsilon_\ell - 1} (SP_\ell\varphi_\ell)\left(\frac{\alpha}{\gamma}\right) = f_k^*(t), \quad k = 1, 2, \ldots, n.$$

Thus, $(P_1\varphi_1,\ P_2\varphi_2, \ldots, P_n\varphi_n)$ is a solution of (3.4).

THEOREM 3.1 *The equation* (3.2) *has solutions in X if and only if the following equation*

$$a^*(t)\varphi_\ell(t) + b^*_{\ell\ell}(t)(S\varphi_\ell)(t) + \frac{b^*_{\ell\ell}(t)}{\varepsilon_\ell - 1}(S\varphi_\ell)\left(\frac{\alpha}{\gamma}\right) = f^*_\ell(t) \tag{3.15}$$

has solutions. Moreover, if $\varphi_\ell(t)$ *is a solution of equation* (3.15) *then equation* (3.2) *has a solution given by formula*

$$\varphi(t) = \frac{f(t) - \sum_{j=1}^m \lambda_j a_j(t) - b(t)(P_\ell S\varphi_\ell)(t)}{a(t)}. \tag{3.16}$$

Proof Suppose that $\varphi \in X$ is a solution of equation (3.2). By Lemma 3.1, $(P_1\varphi, P_2\varphi, \ldots, P_n\varphi)$ is a solution of system (3.4). Hence, $P_\ell\varphi$ is a solution of (3.15).

Conversely, suppose that $\varphi_\ell(t)$ is a solution of (3.15). In this case, system (3.4) has solution $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ determined by the formula

$$\begin{aligned}\varphi_k(t) &= \frac{f^*_k(t) - b^*_{k\ell}(t)(S\varphi_\ell)(t) - ((b^*_{k\ell}(t))/(\varepsilon_\ell - 1))(S\varphi_\ell)(\alpha/\gamma)}{a^*(t)}, \\ &k = 1, 2, \ldots, n, \;\; k \neq \ell.\end{aligned} \tag{3.17}$$

By Lemma 3.2, we have $(P_1\varphi_1, P_2\varphi_2, \ldots, P_n\varphi_n)$ is also a solution of (3.4). Put

$$\varphi = \sum_{k=1}^n P_k\varphi_k. \tag{3.18}$$

It is clear that $P_k\varphi = P_k\varphi_k$. This means that $(P_1\varphi, P_2\varphi, \ldots, P_n\varphi)$ is a solution of (3.4). From Lemma 3.1 it follows that ϕ is a solution of (3.2). Moreover, from (3.17) and (3.18) we get

$$\begin{aligned}\varphi(t) &= \sum_{k=1}^n P_k\varphi_k = \sum_{k=1}^n P_k \frac{f^*_k(t) - b^*_{k\ell}(t)(S\varphi_\ell)(t) - (b^*_{k\ell}(t))/(\varepsilon_\ell - 1)(S\varphi_\ell)(\alpha/\gamma)}{a^*(t)} \\ &= \frac{1}{a^*(t)}\sum_{k=1}^n \left[f^*_k(t) - b^*_{k\ell}(t)(P_\ell S\varphi_\ell)(t) - \frac{b^*_{k\ell}(t)}{\varepsilon_\ell - 1}P_\ell\left((S\varphi_\ell)\left(\frac{\alpha}{\gamma}\right)\right)\right].\end{aligned} \tag{3.19}$$

Substituting (3.9), (3.10), (3.14) into (3.19) we obtain

$$\varphi(t) = \frac{f(t) - \sum_{j=1}^m \lambda_j a_j(t) - b(t)(P_\ell S\varphi_\ell)(t)}{a(t)}.$$

The proof is complete. ■

4. The solvability of equation (3.15)

We set

$$D^+ = \{z \in \mathbb{C} : |z| < 1\}, \quad D^- = \{z \in \mathbb{C} : |z| > 1\}$$

Denote by $H(D^+)$,$H(D^-)$ the sets of the analytic functions in D^+ and D^- respectively. Consider the equation (3.15)

$$a^*(t)\varphi_\ell(t) + b^*_{\ell\ell}(t)(S\varphi_\ell)(t) + \frac{b^*_{\ell\ell}(t)}{\varepsilon_\ell - 1}(S\varphi_\ell)\left(\frac{\alpha}{\gamma}\right) = f^*_\ell(t).$$

Put

$$\Phi_\ell(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi_\ell(\tau)}{\tau - z}\mathrm{d}\tau, \quad z \in \mathbb{C}\backslash\Gamma.$$

According to Sokhotski–Plemelij formula, we have [1]

$$\varphi_\ell(t) = \Phi^+_\ell(t) - \Phi^-_\ell(t), \tag{4.1}$$

$$(S\varphi_\ell)(t) = \Phi^+_\ell(t) + \Phi^-_\ell(t). \tag{4.2}$$

Moreover, $(S\varphi_\ell)(\alpha/\gamma) = 2\Phi_\ell(\alpha/\gamma)$. Put $\lambda_0 = \Phi_\ell(\alpha/\gamma)$. We reduce equation (3.15) to the following boundary problem: find a sectionally analytic function $\Phi_\ell(z)$ ($\Phi(z) = \Phi^+(z)$ for $z \in D^+$, $\Phi(z) = \Phi^-(z)$ for $z \in D^-$) vanishes at infinity, $\Phi_\ell(\alpha/\gamma) = \lambda_0$ and satisfies the following linear relation on Γ

$$\Phi^+_\ell(t) = G(t)\Phi^-_\ell(t) + g(t), \quad t \in \Gamma, \tag{4.3}$$

where

$$G(t) = \frac{a^*(t) - b^*_{\ell\ell}(t)}{a^*(t) + b^*_{\ell\ell}(t)}, \;\; g(t) = \frac{f^*_\ell(t) - \lambda_0((2b^*_{\ell\ell}(t))/(\varepsilon_\ell - 1))}{a^*(t) + b^*_{\ell\ell}(t)}. \tag{4.4}$$

Suppose that $a^*(t) \pm b^*_{\ell\ell}(t)$ are the non-vanishing functions on Γ. Then $G(t)$, $g(t) \in X$ and $G(t) \neq 0$ for any $t \in \Gamma$. Put

$$\begin{aligned} \varkappa = G(t) &= \frac{1}{2\pi i}\int_\Gamma d\ln G(t), \\ \Gamma(z) &= \frac{1}{2\pi i}\int_\Gamma \frac{\ln[\tau^{-\varkappa}G(\tau)]}{\tau - z}\mathrm{d}\tau, \\ X^+(z) = \mathrm{e}^{\Gamma^+(z)}, &\quad X^-(z) = z^{-\varkappa}\mathrm{e}^{\Gamma^-(z)}. \end{aligned} \tag{4.5}$$

Using the results in [2] (p. 16–20) we get the following cases:

(1) $\varkappa \geq 0$. The equation (4.3) has general solution is given by formula

$$\begin{aligned} \Phi_\ell(z) &= X(z)\left[\frac{1}{2\pi i}\int_\Gamma \frac{g(\tau)}{X^+(\tau)}\frac{\mathrm{d}\tau}{\tau - z} + P_{\varkappa-1}(z)\right] \\ &= X(z)\left[\Psi(z) - \lambda_0 B(z) + P_{\varkappa-1}(z)\right], \end{aligned} \tag{4.6}$$

where

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f^*_\ell(\tau)}{X^+(\tau)(a^*(\tau) + b^*_{\ell\ell}(\tau))}\frac{\mathrm{d}\tau}{\tau - z}, \tag{4.7}$$

$$B(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(2b^*_{\ell\ell}(\tau)/\varepsilon_\ell - 1)}{X^+(\tau)(a^*(\tau) + b^*_{\ell\ell}(\tau))}\,\frac{d\tau}{\tau - z}, \tag{4.8}$$

$$P_{\varkappa-1}(z) \equiv 0, \quad \text{if } \varkappa = 0,$$

and

$$P_{\varkappa-1}(z) = p_1 + p_2 z + \cdots + p_\varkappa z^{\varkappa-1}, \quad \text{if } \varkappa \geq 1, \tag{4.9}$$

which is a polynomial of degree $\varkappa - 1$ with arbitrary complex coefficients. The function $\Phi_\ell(z)$ determined in (4.6) is a solution of problem (4.3) if $\Phi_\ell(\alpha/\gamma) = \lambda_0$, that is

$$X\left(\frac{\alpha}{\gamma}\right)\left[\Psi\left(\frac{\alpha}{\gamma}\right) - \lambda_0 B\left(\frac{\alpha}{\gamma}\right) + P_{\varkappa-1}\left(\frac{\alpha}{\gamma}\right)\right] = \lambda_0.$$

This implies

$$\lambda_0\left[1 + X\left(\frac{\alpha}{\gamma}\right)B\left(\frac{\alpha}{\gamma}\right)\right] = X\left(\frac{\alpha}{\gamma}\right)\left[\Psi\left(\frac{\alpha}{\gamma}\right) + P_{\varkappa-1}\left(\frac{\alpha}{\gamma}\right)\right]. \tag{4.10}$$

(i) If $1 + X(\alpha/\gamma)B(\alpha/\gamma) \neq 0$: from (4.10), we get

$$\lambda_0 = \frac{X(\alpha/\gamma)[\Psi(\alpha/\gamma) + P_{\varkappa-1}(\alpha/\gamma)]}{1 + X(\alpha/\gamma)B(\alpha/\gamma)}. \tag{4.11}$$

In this case, the general solution of problem (4.3) is given by formula

$$\Phi_\ell(z) = X(z)\left[\Psi(z) - \frac{X(\alpha/\gamma)[\Psi(\alpha/\gamma) + P_{\varkappa-1}(\alpha/\gamma)]}{1 + X(\alpha/\gamma)B(\alpha/\gamma)}B(z) + P_{\varkappa-1}(z)\right], \tag{4.12}$$

where $X(z)$, $\Psi(z)$, $B(z)$ are determined by (4.5), (4.7), (4.8) and $P_{\varkappa-1}(z)$ is a polynomial of degree $\varkappa - 1$ with arbitrary complex coefficients.

(ii) If $1 + X(\alpha/\gamma)B(\alpha/\gamma) = 0$: from (4.10) we get

$$\Psi\left(\frac{\alpha}{\gamma}\right) + P_{\varkappa-1}\left(\frac{\alpha}{\gamma}\right) = 0. \tag{4.13}$$

Then, the general solution of problem (4.3) is given by formula

$$\Phi_\ell(z) = X(z)[\Psi(z) - \lambda_0 B(z) + P_{\varkappa-1}(z)], \tag{4.14}$$

where $X(z)$, $\Psi(z)$, $B(z)$ are determined by (4.5), (4.7), (4.8), λ_0 is arbitrary and $P_{\varkappa-1}(z)$ is a polynomial of degree $\varkappa - 1$ with complex coefficients satisfying condition (4.13).

(2) $\varkappa < 0$. The necessary condition for the problem (4.3) to be solvable is that

$$\int_\Gamma \frac{g(\tau)}{X^+(\tau)}\tau^{\varkappa-1}d\tau = 0, \; k = 1, \ldots, -\varkappa.$$

This condition can be written as follows

$$\int_\Gamma \frac{f^*_\ell(\tau)\tau^{\varkappa-1}}{X^+(\tau)(a^*(\tau)+b^*_{\ell\ell}(\tau))}\mathrm{d}\tau = \lambda_0 \int_\Gamma \frac{((2b^*_{\ell\ell}(\tau))/(\varepsilon_\ell - 1))\tau^{k-1}}{X^+(\tau)(a^*(\tau)+b^*_{\ell\ell}(\tau))}\mathrm{d}\tau, \quad k=1,\ldots,-\varkappa. \tag{4.15}$$

In this case, we have $P_{\varkappa-1}(z)\equiv 0$. So we receive

(i) If $1+X(\alpha/\gamma)B(\alpha/\gamma)\neq 0$: from (4.11) we get

$$\lambda_0 = \frac{X(\alpha/\gamma)\Psi(\alpha/\gamma)}{1+X(\alpha/\gamma)B(\alpha/\gamma)}.$$

Hence, (4.15) becomes the following condition

$$\int_\Gamma \frac{f^*_\ell(\tau)\tau^{k-1}}{X^+(\tau)(a^*(\tau)+b^*_{\ell\ell}(\tau))}\mathrm{d}\tau = \frac{X(\alpha/\gamma)\Psi(\alpha/\gamma)}{1+X(\alpha/\gamma)B(\alpha/\gamma)} \int_\Gamma \frac{(2b^*_{\ell\ell}(\tau)/\varepsilon_\ell - 1)\tau^{k-1}}{X^+(\tau)(a^*(\tau)+b^*_{\ell\ell}(\tau))}\mathrm{d}\tau, \quad k=1,\ldots,-\varkappa. \tag{4.16}$$

If the condition (4.16) is satisfied then the solution of the problem (4.3) is given by formula

$$\Phi_\ell(z) = X(z)\left[\Psi(z) - \frac{X(\alpha/\gamma)\Psi(\alpha/\gamma)}{1+X(\alpha/\gamma)B(\alpha/\gamma)}B(z)\right], \tag{4.17}$$

where $X(z)$, $\Psi(z)$, $B(z)$ are determined by (4.5), (4.7), (4.8).

(ii) If $1+X(\alpha/\gamma)B(\alpha/\gamma)=0$: from (4.13) we get

$$\Psi\left(\frac{\alpha}{\gamma}\right) = 0. \tag{4.18}$$

If the conditions (4.15) and (4.18) are satisfied then the solution of the problem (4.3) is given by formula

$$\Phi_\ell(z) = X(z)\left[\Psi(z) - \lambda_0 B(z)\right], \tag{4.19}$$

where $X(z)$, $\Psi(z)$, $B(z)$ are determined by (4.5), (4.7), (4.8) and λ_0 is determined from condition (4.15).

Now we can formulate the main results about solutions of the equation (3.15) in the following form

THEOREM 4.1 *Suppose that the functions $a^*(t)\pm b^*_{\ell\ell}(t)$ does not vanish on Γ.*

(1) *If* $1+X(\alpha/\gamma)B(\alpha/\gamma)\neq 0$ and $\varkappa\geq 0$ *then equation* (3.15) *has solutions φ_ℓ which satisfy the following formula*

$$S\varphi_\ell(t) = X^+(t)\left[\Psi^+(t) - \frac{X(\alpha/\gamma)[\Psi(\alpha/\gamma)+P_{\varkappa-1}(\alpha/\gamma)]}{1+X(\alpha/\gamma)B(\alpha/\gamma)}B^+(t) + P_{\varkappa-1}(t)\right] + X^-(t)\left[\Psi^-(t) - \frac{X(\alpha/\gamma)[\Psi(\alpha/\gamma)+P_{\varkappa-1}(\alpha/\gamma)]}{1+X(\alpha/\gamma)B(\alpha/\gamma)}B^-(t) + P_{\varkappa-1}(t)\right], \tag{4.20}$$

where $X(z)$, $\Psi(z)$, $B(z)$ *are determined by* (4.5), (4.7), (4.8) *and* $P_{\varkappa-1}(z)$ *is a polynomial of degree* $\varkappa - 1$ *with arbitrary complex coefficients.*

(2) *If* $1 + X(\alpha/\gamma)B(\alpha/\gamma) \neq 0$ *and* $\varkappa < 0$ *then equation* (3.15) *is solvable if the condition* (4.16) *is satisfied. In this case, equation* (3.15) *has unique solution which satisfies the formula* (4.20), *where* $P_{\varkappa-1}(z) \equiv 0$.

(3) *If* $1 + X\alpha/\gamma)B(\alpha/\gamma) = 0$ *and* $\varkappa \geq 0$ *then equation* (3.15) *has solutions* φ_ℓ *which satisfy the following formula*

$$S\varphi_\ell(t) = X^+(t)\big[\Psi^+(t) - \lambda_0 B^+(t) + P_{\varkappa-1}(t)\big] + X^-(t)[\Psi^-(t) - \lambda_0 B^-(t) + P_{\varkappa-1}(t)], \tag{4.21}$$

where $X(z)$, $\Psi(z)$, $B(z)$ are determined by (4.5), (4.7), (4.8), λ_0 *is arbitrary and* $P_{\varkappa-1}(z)$ *is a polynomial of degree* $\varkappa - 1$ *with complex coefficients satisfying the condition* (4.13).

(4) *If* $1 + X(\alpha/\gamma)B(\alpha/\gamma) = 0$ *and* $\varkappa < 0$ *then the equation* (3.15) *is solvable if the condition* (4.15) *and* (4.18) *are satisfied. In this case, equation* (3.15) *has unique solution which satisfies the formula* (4.21), *where* $P_{\varkappa-1}(z) \equiv 0$ *and* λ_0 *is determined from the condition* (4.15).

Proof (1) From assumption it follows that the problem (4.3) has a solution $\Phi_\ell(z)$ determined by (4.12). Therefore, equation (3.15) has a solution $\varphi_\ell(t)$ determined by (4.1). Moreover, from (4.2) we get

$$\begin{aligned} S\varphi_\ell(t) &= \Phi_\ell^+(t) + \Phi_\ell^-(t) \\ &= X^+(t)\left[\Psi^+(t) - \frac{X(\alpha/\gamma)\,[\Psi(\alpha/\gamma) + P_{\varkappa-1}(\alpha/\gamma)]}{1 + X(\alpha/\gamma)B(\alpha/\gamma)} B^+(t) + P_{\varkappa-1}(t)\right] \\ &\quad + X^-(t)\left[\Psi^-(t) - \frac{X(\alpha/\gamma)\,[\Psi(\alpha/\gamma) + P_{\varkappa-1}(\alpha/\gamma)]}{1 + X(\alpha/\gamma)B(\alpha/\gamma)} B^-(t) + P_{\varkappa-1}(t)\right]. \end{aligned}$$

Similarly, the cases (2), (3), (4) can be proved. ■

5. The solvability of equation (3.1)

Theorems 3.1 and 4.1 show that if $a^*(t) \pm b^*_{\ell\ell}(t) \neq 0$ on Γ then equation (3.2) is solvable in a closed form. In this section, we study which solutions of (3.2) will be the solution of (3.1), i.e., the solutions of (3.2) need to satisfy the condition (3.3). Consider the following cases:

(1) $1 + X(\alpha/\gamma)B(\alpha/\gamma) \neq 0$, $\varkappa \geq 0$. By using Theorems 3.1 and 4.1, we have solutions of (3.2) given by the following formula

$$\varphi(t) = \frac{f(t) - \sum_{j=1}^{m} \lambda_j a_j(t) - b(t)(P_\ell S\varphi_\ell)(t)}{a(t)}, \tag{5.1}$$

where $S\varphi_\ell(t)$ is determined by (4.20). From (3.5) and (4.7) we get

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(1/n)\sum_{j=1}^{n}\varepsilon_\ell^{n-1-j}\big[f(\omega^{j+1}(\tau)) - \sum_{\nu=1}^{m}\lambda_\nu a_\nu(\omega^{j+1}(\tau))\big]\prod_{\substack{\mu=1\\ \mu\neq j}}^{n} a(\omega^{\mu+1}(\tau))}{X^+(\tau)(a^*(\tau)+b^*_{\ell\ell}(\tau))}\frac{d\tau}{\tau - z}$$

$$= \Psi_1(z) - \sum_{\nu=1}^{m}\lambda_\nu A_\nu(z), \tag{5.2}$$

where

$$\Psi_1(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(1/n)\sum_{j=1}^{n}\varepsilon_\ell^{n-1-j} f(\omega^{j+1}(\tau))\prod_{\substack{\mu=1\\ \mu\neq j}}^{n} a(\omega^{\mu+1}(\tau))}{X^+(\tau)(a^*(\tau)+b^*_{\ell\ell}(\tau))}\frac{d\tau}{\tau - z}, \tag{5.3}$$

$$A_\nu(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(1/n)\sum_{j=1}^{n}\varepsilon_\ell^{n-1-j} a_\nu(\omega^{j+1}(\tau))\prod_{\substack{\mu=1\\ \mu\neq j}}^{n} a(\omega^{\mu+1}(\tau))}{X^+(\tau)(a^*(\tau)+b^*_{\ell\ell}(\tau))}\frac{d\tau}{\tau - z}. \tag{5.4}$$

Substituting (4.9), (5.2) into (4.20) we find

$$S\varphi_\ell(t) = [X^+(t)\Psi_1^+(t) + X^-(t)\Psi_1^-(t)] - \sum_{j=1}^{m}\lambda_j[X^+(t)A_j^+(t) + X^-(t)A_j^-(t)]$$

$$- \frac{X(\alpha/\gamma)\big[\Psi_1(\alpha/\gamma) - \sum_{j=1}^{m}\lambda_j A_j(\alpha/\gamma) + \sum_{j=1}^{\varkappa} p_j(\alpha/\gamma)^{j-1}\big]}{1 + X(\alpha/\gamma)B(\alpha/\gamma)}[X^+(t)B^+(t) + X^-(t)B^-(t)]$$

$$+ \sum_{j=1}^{\varkappa} p_j t^{j-1}[X^+(t) + X^-(t)].$$

Since we can rewrite (5.1) in the form

$$\varphi(t) = \frac{f(t) - b(t)P_\ell[X^+(t)\Psi_1{}^+(t) + X^-(t)\Psi_1^-(t)]}{a(t)}$$

$$- \sum_{j=1}^{m}\lambda_j\frac{a_j(t) - b(t)P_\ell[X^+(t)A_j^+(t) + X^-(t)A_j^-(t)]}{a(t)}$$

$$+ \frac{X(\alpha/\gamma)\big[\Psi_1(\alpha/\gamma) - \sum_{j=1}^{m}\lambda_j A_j(\alpha/\gamma) + \sum_{j=1}^{\varkappa} p_j(\alpha/\gamma)^{j-1}\big]}{1 + X(\alpha/\gamma)B(\alpha/\gamma)}$$

$$\times \frac{b(t)P_\ell[X^+(t)B^+(t) + X^-(t)B^-(t)]}{a(t)}$$

$$- \sum_{j=1}^{\varkappa} p_j\frac{b(t)P_\ell\big[t^{j-1}[X^+(t) + X^-(t)]\big]}{a(t)}, \tag{5.5}$$

where $X(z)$, $B(z)$, $\Psi_1(z)$, $A_1(z), \ldots, A_m(z)$ are determined by (4.5), (4.8), (5.3), (5.4), and $p_1, \ldots, p_\varkappa$ are arbitrary. The function φ is a solution of the equation (3.1) if it satisfies the condition (3.3), that is

$$M_{b_k}\varphi = \lambda_k, \quad k = 1, \ldots, m.$$

Substituting (5.5) into the last condition, we obtain

$$\begin{aligned}\lambda_k &= d_k - \sum_{j=1}^{m} e_{kj}\lambda_j + \left[\Psi_1\left(\frac{\alpha}{\gamma}\right) - \sum_{j=1}^{m}\lambda_j A_j\left(\frac{\alpha}{\gamma}\right) + \sum_{j=1}^{\varkappa} p_j\left(\frac{\alpha}{\gamma}\right)^{j-1}\right] f_k - \sum_{j=1}^{\varkappa} g_{kj}p_j \\ &= \left[d_k + \Psi_1\left(\frac{\alpha}{\gamma}\right)f_k\right] - \sum_{j=1}^{m}\left[e_{kj} + f_k A_j\left(\frac{\alpha}{\gamma}\right)\right]\lambda_j - \sum_{j=1}^{\varkappa}\left[g_{kj} - f_k\left(\frac{\alpha}{\gamma}\right)^{j-1}\right]p_j, \\ k &= 1, 2, \ldots, m, \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} d_k &= M_{b_k}\left(\frac{f(t) - b(t)P_\ell[X^+(t)\Psi_1^+(t) + X^-(t)\Psi_1^-(t)]}{a(t)}\right), \\ e_{kj} &= M_{b_k}\left(\frac{a_j(t) - b(t)P_\ell[X^+(t)A_j^+(t) + X^-(t)A_j^-(t)]}{a(t)}\right), \\ f_k &= M_{b_k}\left(\frac{X(\alpha/\gamma)}{1 + X(\alpha/\gamma)B(\alpha/\gamma)}\ \frac{b(t)P_\ell[X^+(t)B^+(t) + X^-(t)B^-(t)]}{a(t)}\right), \\ g_{kj} &= M_{b_k}\left(\frac{b(t)P_\ell\left[t^{j-1}[X^+(t) + X^-(t)]\right]}{a(t)}\right). \end{aligned} \tag{5.7}$$

Put

$$\lambda = \begin{pmatrix}\lambda_1 \\ \vdots \\ \lambda_m\end{pmatrix}_{m\times 1}, \quad P = \begin{pmatrix}p_1 \\ \vdots \\ p_\varkappa\end{pmatrix}_{\varkappa\times 1}, \quad D = \begin{pmatrix} d_1 + \Psi_1\left(\frac{\alpha}{\gamma}\right)f_1 \\ \vdots \\ d_m + \Psi_1\left(\frac{\alpha}{\gamma}\right)f_m\end{pmatrix}_{m\times 1},$$

$$E = \begin{pmatrix} e_{11} + f_1A_1\left(\frac{\alpha}{\gamma}\right) & \cdots & e_{1m} + f_1A_m\left(\frac{\alpha}{\gamma}\right) \\ \vdots & \ddots & \vdots \\ e_{m1} + f_mA_1\left(\frac{\alpha}{\gamma}\right) & \cdots & e_{mm} + f_mA_m\left(\frac{\alpha}{\gamma}\right)\end{pmatrix}_{m\times m}, \tag{5.8}$$

$$G = \begin{pmatrix} g_{11} - f_1\left(\frac{\alpha}{\gamma}\right)^0 & \cdots & g_{1\varkappa} - f_1\left(\frac{\alpha}{\gamma}\right)^{\varkappa-1} \\ \vdots & \ddots & \vdots \\ g_{m1} - f_m\left(\frac{\alpha}{\gamma}\right)^0 & \cdots & g_{m\varkappa} - f_m\left(\frac{\alpha}{\gamma}\right)^{\varkappa-1}\end{pmatrix}_{m\times\varkappa.}$$

Now we write (5.6) in the form of matrix condition

$$(I + E)\lambda = D - GP, \tag{5.9}$$

where I is the unit matrix. So we can formulate that the function φ determined by (5.5) is a solution of (3.1) if and only if $(\lambda_1, \ldots, \lambda_m)$ satisfy the condition (5.9).

(2) $1 + X(\alpha/\gamma)B(\alpha/\gamma) \neq 0, \varkappa < 0$. From Theorems 3.1 and 4.1 it follows that the equation (3.2) has solutions if and only if the condition (4.16) satisfied. If this is in case, then $P_{\varkappa-1} \equiv 0$. So, the solutions of (3.2) are given by as follows

$$\varphi(t) = \frac{f(t) - b(t)P_\ell[X^+(t)\Psi_1^+(t) + X^-(t)\Psi_1^-(t)]}{a(t)} - \sum_{j=1}^{m} \lambda_j \frac{a_j(t) - b(t)P_\ell[X^+(t)A_j^+(t) + X^-(t)A_j^-(t)]}{a(t)} + \frac{X(\alpha/\gamma)\left[\Psi_1(\alpha/\gamma) - \sum_{j=1}^m \lambda_j A_j(\alpha/\gamma)\right]}{1 + X(\alpha/\gamma)B(\alpha/\gamma)} \frac{b(t)P_\ell[X^+(t)B^+(t) + X^-(t)B^-(t)]}{a(t)}. \quad (5.10)$$

Therefore, the function ϕ determined by (5.10) is a solution of the equation (3.1) if and only if $(\lambda_1, \ldots, \lambda_m)$ satisfy the following matrix condition

$$(I + E)\lambda = D, \quad (5.11)$$

where E and D are determined by (5.8). On the other hand, substituting (3.5), (5.2) into (4.16) we get

$$d'_k - \sum_{\nu=1}^{m} e'_{k\nu}\lambda_\nu = \Psi_1\left(\frac{\alpha}{\gamma}\right)f'_k - \sum_{j=1}^{m} f'_k A_j\left(\frac{\alpha}{\gamma}\right)\lambda_j, \quad k = 1, 2, \ldots, -\varkappa, \quad (5.12)$$

where

$$d'_k = \int_\Gamma \frac{(1/n)\sum_{j=1}^n \varepsilon_\ell^{n-1-j} f(\omega^{j+1}(\tau)) \prod_{\substack{\mu=1 \\ \mu\neq j}}^n a(\omega^{\mu+1}(\tau))}{X^+(\tau)(a^*(\tau) + b^*_{\ell\ell}(\tau))} \tau^{k-1} \mathrm{d}\tau$$

$$e'_{k\nu} = \int_\Gamma \frac{(1/n)\sum_{j=1}^n \varepsilon_\ell^{n-1-j} a_\nu(\omega^{j+1}(\tau)) \prod_{\substack{\mu=1 \\ \mu\neq j}}^n a(\omega^{\mu+1}(\tau))}{X^+(\tau)(a^*(\tau) + b^*_{\ell\ell}(\tau))} \tau^{k-1} \mathrm{d}\tau$$

$$f'_k = \frac{X(\alpha/\gamma)}{1 + X(\alpha/\gamma)B(\alpha/\gamma)} \int_\Gamma \frac{((2b^*_{\ell\ell}(\tau))/(\varepsilon_\ell - 1))}{X^+(\tau)(a^*(\tau) + b^*_{\ell\ell}(\tau))} \tau^{k-1} \mathrm{d}\tau. \quad (5.13)$$

Put

$$D' = \begin{pmatrix} d'_1 - \Psi_1\left(\frac{\alpha}{\gamma}\right)f'_1 \\ \vdots \\ d'_{-\varkappa} - \Psi_1\left(\frac{\alpha}{\gamma}\right)f'_{-\varkappa} \end{pmatrix}_{-\varkappa\times 1},$$

$$E' = \begin{pmatrix} e'_{11} - f'_1 A_1\left(\frac{\alpha}{\gamma}\right) & \cdots & e'_{1m} - f'_1 A_m\left(\frac{\alpha}{\gamma}\right) \\ \vdots & \ddots & \vdots \\ e'_{-\varkappa 1} - f'_{-\varkappa} A_1\left(\frac{\alpha}{\gamma}\right) & \cdots & e'_{-\varkappa m} - f'_{-\varkappa} A_m\left(\frac{\alpha}{\gamma}\right) \end{pmatrix}_{-\varkappa\times m}. \quad (5.14)$$

We write (5.12) in the form of matrix condition

$$E'\lambda = D'. \tag{5.15}$$

Combining (5.11) and (5.15) we can say that the function ϕ determined by (5.10) is a solution of (3.1) if and only if $(\lambda_1, \ldots, \lambda_m)$ satisfy the following matrix condition

$$\begin{pmatrix} I+E \\ E' \end{pmatrix}_{(m-\varkappa)\times m} \lambda = \begin{pmatrix} D' \\ D \end{pmatrix}_{(m-\varkappa)\times 1}. \tag{5.16}$$

(3) $1 + X(\alpha/\gamma)B(\alpha/\gamma) = 0, \varkappa \geq 0$. Then the solutions of the equation (3.2) are given by the following formula

$$\begin{aligned}\varphi(t) =& \frac{f(t) - b(t)P_\ell[X^+(t)\Psi_1^+(t) + X^-(t)\Psi_1^-(t)]}{a(t)} \\ &- \sum_{j=1}^{m} \lambda_j \frac{a_j(t) - b(t)P_\ell[X^+(t)A_j^+(t) + X^-(t)A_j^-(t)]}{a(t)} \\ &+ \lambda_0 \frac{b(t)P_\ell[X^+(t)B^+(t) + X^-(t)B^-(t)]}{a(t)} \\ &- \sum_{j=1}^{\varkappa} p_j \frac{b(t)P_\ell\left[t^{j-1}[X^+(t) + X^-(t)]\right]}{a(t)},\end{aligned} \tag{5.17}$$

where $X(z)$, $B(z)$, $\Psi_1(z)$, $A_1(z), \ldots, A_m(z)$ are determined by (4.5), (4.8), (5.3), (5.4), λ_0 is an arbitrary complex number and $p_1, \ldots, p_\varkappa$ satisfy the condition (4.13). Substituting (5.2) in (4.13) we obtain

$$\Psi_1\left(\frac{\alpha}{\gamma}\right) - \sum_{j=1}^{m} \lambda_j A_j\left(\frac{\alpha}{\gamma}\right) + \sum_{j=1}^{\varkappa} p_j \left(\frac{\alpha}{\gamma}\right)^{j-1} = 0. \tag{5.18}$$

The function φ is a solution of the equation (3.1) if it satisfies the condition (3.3). Substituting (5.17) into (3.3) we get

$$\begin{aligned}\lambda_k &= d_k - \sum_{j=1}^{m} e_{kj}\lambda_j + \lambda_0 f_k - \sum_{j=1}^{\varkappa} g_{kj} p_j \\ &= d_k + \lambda_0 f_k - \sum_{j=1}^{m} e_{kj}\lambda_j - \sum_{j=1}^{\varkappa} g_{kj} p_j, \quad k = 1, 2, \ldots, m,\end{aligned} \tag{5.19}$$

where d_k, e_{kj}, f_k, g_{kj} are determined by (5.7). Put

$$\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}_{m\times 1}, \quad D = \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix}_{m\times 1}, \quad F = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}_{m\times 1}, \quad P = \begin{pmatrix} p_1 \\ \vdots \\ p_\varkappa \end{pmatrix}_{\varkappa\times 1},$$

$$E = \begin{pmatrix} e_{11} & \cdots & e_{1m} \\ \vdots & \ddots & \vdots \\ e_{m1} & \cdots & e_{mm} \end{pmatrix}_{m\times m}, \quad G = \begin{pmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & \ddots & \vdots \\ g_{mj} & \cdots & g_{m\varkappa} \end{pmatrix}_{m\times\varkappa}. \tag{5.20}$$

We write (5.19) in the form of matrix condition

$$(I+E)\lambda = D + \lambda_0 F - GP. \tag{5.21}$$

Combining (5.18) and (5.21) we can say that the function φ determined by (5.17) is a solution of (3.1) if and only if $(\lambda_1, \dots, \lambda_m)$ satisfy the following matrix condition

$$(\overline{I+E})\lambda = \overline{D} + \lambda_0 \overline{F} - \overline{G}P, \tag{5.22}$$

where

$$\overline{I+E} = \begin{pmatrix} I+E \\ A_1\left(\frac{\alpha}{\gamma}\right) \cdots A_m\left(\frac{\alpha}{\gamma}\right) \end{pmatrix}_{(m+1)\times m}, \quad \overline{G} = \begin{pmatrix} G \\ -\left(\frac{\alpha}{\gamma}\right)^0 \cdots -\left(\frac{\alpha}{\gamma}\right)^{\varkappa-1} \end{pmatrix}_{(m+1)\times\varkappa},$$
$$\overline{D} = \begin{pmatrix} D \\ \Psi_1\left(\frac{\alpha}{\gamma}\right) \end{pmatrix}_{(m+1)\times 1}, \quad \overline{F} = \begin{pmatrix} F \\ 0 \end{pmatrix}_{(m+1)\times 1}. \tag{5.23}$$

(4) $1 + X(\alpha/\gamma)B(\alpha/\gamma) = 0, \varkappa < 0$. By Theorems 3.1 and 4.1 the equation (3.2) has solutions if and only if the condition (4.15) and the conditions (4.18) are satisfied. Then we have $\varkappa > 0$ with $P_{\varkappa-1} \equiv 0$. So the solutions of (3.2) are given by the following formula

$$\begin{aligned}\varphi(t) = &\frac{f(t) - b(t)P_\ell[X^+(t)\Psi_1^+(t) + X^-(t)\Psi_1^-(t)]}{a(t)} \\ &- \sum_{j=1}^m \lambda_j \frac{a_j(t) - b(t)P_\ell[X^+(t)A_j^+(t) + X^-(t)A_j^-(t)]}{a(t)} \\ &+ \lambda_0 \frac{b(t)P_\ell[X^+(t)B^+(t) + X^-(t)B^-(t)]}{a(t)}.\end{aligned} \tag{5.24}$$

The function φ determined by (5.24) is a solution of the equation (3.1) if and only if $(\lambda_1, \dots, \lambda_m)$ satisfy the following matrix condition

$$(\overline{I+E})\lambda = \overline{D} + \lambda_0 \overline{F}, \tag{5.25}$$

where $\overline{I+E}$, $\overline{D}$, $\overline{F}$ are determined by (5.23). On the other hand, (4.15) is equivalent to the condition

$$d'_k - \sum_{j=1}^m e'_{kj}\lambda_j = \lambda_0 f'_k, \quad k = 1, 2, \dots, -\varkappa, \tag{5.26}$$

where d'_K, e'_{kj}, f'_k, are determined by (5.13).

(5) Put

$$D' = \begin{pmatrix} d'_1 \\ \vdots \\ d'_{-\varkappa} \end{pmatrix}_{-\varkappa\times 1}, \quad F' = \begin{pmatrix} -f'_1 \\ \vdots \\ -f'_{-\varkappa} \end{pmatrix}_{-\varkappa\times 1}, \quad E' = \begin{pmatrix} e'_{11} & \cdots & e'_{1m} \\ \vdots & \ddots & \vdots \\ e'_{-\varkappa 1} & \cdots & e'_{-\varkappa m} \end{pmatrix}_{-\varkappa\times m}. \tag{5.27}$$

We write (5.26) in the form of matrix condition

$$E'\lambda = D' + \lambda_0 F'. \tag{5.28}$$

Combining (5.25) and (5.28) we can say that the function φ determined by (5.24) is a solution of (3.1) if and only if $(\lambda_1, \dots, \lambda_m)$ satisfy the following matrix condition

$$\begin{pmatrix} \overline{I+E} \\ E' \end{pmatrix}_{(m+1-\varkappa)\times m} \lambda = \begin{pmatrix} \overline{D} \\ D' \end{pmatrix}_{(m+1-\varkappa)\times 1} + \lambda_0 \begin{pmatrix} \overline{F} \\ F' \end{pmatrix}_{(m+1-\varkappa)\times 1.} \tag{5.29}$$

Remark 5.1 Among the matrices $\{D, E, G, F, D', E', F', \overline{D}, \overline{(I+E)}, \overline{F}\}$, there are three matrices $D, D', \overline{D}$ which depending on f(t), the remaining ones are completely determined by $a(t)$, $b(t)$, $a_1(t), \dots, a_m(t)$, $b_1(t), \dots, b_m(t)$.

THEOREM 5.1 *Suppose* $a^{*2}(t) - b^{*2}_{\ell\ell}(t) \neq 0$ *for any* $t \in \Gamma$.

(1) $1 + X(\alpha/\gamma)B(\alpha/\gamma) \neq 0,\ \varkappa \geq 0$. Put

$$r = \operatorname{rank}\,((I+E)\,G\,)_{m\times(m+\varkappa)},$$

where E, G *are determined by* (5.8). *Then the equation* (3.1) *is solvable if and only if the matrix* D *determined by* (5.8) *satisfies the matrix condition*

$$\operatorname{rank}((I+E)\,GD)_{m\times(m+\varkappa+1)} = r.$$

If this is in case, the solutions of the equation (3.1) *are given by the formula* (5.5), *where* $(\lambda_1, \dots, \lambda_m,\ p_1, \dots, p_\varkappa)$ *satisfy* (5.9). *Moreover, we can choose* $m + \varkappa - r$ *coefficients in* $\{\lambda_1, \dots, \lambda_m, p_1, \dots, p_\varkappa\}$ *which are arbitrary so that* $\phi(t)$ *is uniquely determined by these coefficients. In particular, if* $r = m$ *then the equation* (3.1) *is solvable for any function* $f(t)$.

(2) $1 + X(\alpha/\gamma)B(\alpha/\gamma) \neq 0,\ \varkappa < 0$. *Put*

$$r = \operatorname{rank}\begin{pmatrix} I+E \\ E' \end{pmatrix}_{(m-\varkappa)\times m},$$

where E, E' *are determined by* (5.8), (5.14). *Then the equation* (3.1) *is solvable if and only if the function* $f(t)$ *determines* D *and* D' *by the formulas* (5.8), (5.14) *which satisfy the following matrix condition*

$$\operatorname{rank}\begin{pmatrix} I+E & D \\ E' & D' \end{pmatrix}_{(m-\varkappa)\times(m+1)} = r. \tag{5.30}$$

If this is in case, the solutions of the equation (3.1) *are given by the formula* (5.10), *where* $(\lambda_1, \dots, \lambda_m)$ satisfy (5.16). *In particular, if* $r = m$ *and the condition* (5.30) *is satisfied then the equation* (3.1) *has unique solution.*

(3) $1 + X(\alpha/\gamma)B(\alpha/\gamma) = 0,\ \varkappa \geq 0$. *Put*

$$r = \operatorname{rank}\left(\overline{I+E}\ \ \overline{F}\ \ \overline{G}\right)_{(m+1)\times(m+1+\varkappa)},$$

where $\overline{I+E}$, $\overline{F}$, $\overline{G}$ *are determined by* (5.23). *The equation* (3.1) *is solvable if and only if the matrix* $\overline{D}$ *determined by formulas* (5.23) *satisfies the matrix condition*

$$\operatorname{rank}\Big(\overline{I+E}\ \ \overline{F}\ \ \overline{G}\ \ \overline{D}\Big)_{(m+1)\times(m+2+\varkappa)} = r.$$

If the above condition satisfied then solutions of the equation (3.1) *are given by the formula* (5.17), *where* $(\lambda_0,\ldots,\lambda_m,p_1,\ldots,p_\varkappa)$ *satisfy* (5.22). *Moreover, we can choose* $m+1+\varkappa-r$ *coefficients in* $\{\lambda_0,\ldots,\lambda_m,p_1,\ldots,p_\varkappa\}$ *which are arbitrary so that* $\varphi(t)$ *is uniquely determined by these coefficients. In particular, if* $r=m+1$ *then the equation* (3.1) *is solvable for any function* $f(t)$.

(4) $1+X(\alpha/\gamma)B(\alpha/\gamma)=0$, $\varkappa<0$. Put

$$r=\operatorname{rank}\begin{pmatrix}\overline{I+E} & \overline{F}\\ E' & F'\end{pmatrix}_{(m+1-\varkappa)\times(m+1)},$$

where $\overline{I+E}$, $\overline{F}$, E', F' *are determined by* (5.23) *and* (5.27). *Then the equation* (3.1) *is solvable if and only if the matrix* $\overline{D}$ *determined by the formulas* (5.23), (5.27) *satisfies the matrix condition*

$$\operatorname{rank}\begin{pmatrix}\overline{I+E} & \overline{F} & \overline{D}\\ E' & F' & D'\end{pmatrix}_{(m+1-\varkappa)\times(m+2)} = r. \qquad (5.31)$$

If the above condition satisfied then solutions of the equation (3.1) *are given by the formula* (5.24), *where* $(\lambda_0,\lambda_1,\ldots,\lambda_m)$ *satisfy* (5.29). *In particular, if* $r=m+1$ *and the condition* (5.31) *is satisfied then the equation* (3.1) *has unique solution.*

Proof (1) From assumption it follows that the equation (3.1) has solutions if and only if there exists $(\lambda_1,\ldots,\lambda_m)$ and $(p_1,\ldots,p_\varkappa)$ which satisfy (5.9). We can rewrite (5.9) in the form

$$\Big((I+E)\ \ G\Big)_{m\times(m+\varkappa)}\begin{pmatrix}\lambda\\ P\end{pmatrix}_{(m+\varkappa)\times 1} = D.$$

Therefore $\binom{\lambda}{P}$ is a solution of the following equation

$$\Big((I+E)\ \ G\Big)X = D. \qquad (5.32)$$

It follows that the necessary and sufficient condition for which the equation (3.1) has solutions is that the equation (5.32) has solutions in $\mathbb{C}^{m+\varkappa}$. Since

$$\operatorname{rank}((I+E)GD)=\operatorname{rank}((I+E)G)=r.$$

If this is in case, then by using (5.32) we can express r coefficients in $\{\lambda_1,\ldots,\lambda_m, p_1,\ldots,p_\varkappa\}$ by $m+\varkappa-r$ remaining ones. In particular, if $r=m$ then the equation (5.32) has solutions with any D. Therefore the equation (3.1) is solvable with any $f(t)$.

Similarly, the cases (2), (3), (4) can be proved. ∎

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